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PROXIMATE GOL'DBERG ORDER AND TYPE OF A MULTIPLE ENTIRE DIRICHLET SERIES

MONALISA MIDDYA
Department of Mathematics,
St. Xavier's College, 30, Mother Teresa Sarani, Kolkata 700016, India

Abstract

In this paper, we have introduced the concept of Proximate Gol'dberg order and Proximate Gol'dberg type of a multiple entire Dirichlet series. We have given some examples of proximate Gol'dberg order and type and also proved theorem to construct a new proximate Gol'dberg order with the help of Gol'dberg order and existing proximate Gol'dberg order of the function.

1. Introduction

1.1 Notations

For $s = (s_1, s_2, ..., s_n), w = (w_1, w_2, ..., w_n) \in \mathcal{C}^n$ and $\alpha \in \mathcal{C}$, we define

- $s = w$ if and only if $s_i = w_i$,

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- $s+w=(s_1+w_1,s_2+w_2,...,s_n+w_n)$,
- $\alpha s=(\alpha s_1,\alpha s_2,...,\alpha s_n)$,
- $s.w=s_1w_1+s_2w_2+...+s_nw_n$,
- $\mid s\mid=(|s_1|^2+|s_2|^2+...+|s_n|^2)^{\frac{1}{2}}$.
- $s+R=(s_1+R,s_2+R,...,s_n+R)$, for $R\in\mathcal{R}$.

We also define

- $\lambda_{n,m}=(\lambda_{1m_1},\lambda_{2m_2},...,\lambda_{nm_n})\in\mathcal{R}^{+^n}$ where $\mathcal{R}^{+^n}=\{x:x\in\mathcal{R}^n,x_i\geq 0\}$.
- $s.\lambda_{n,m}=s_1\lambda_{1m_1}+s_2\lambda_{2m_2}+...+s_n\lambda_{nm_n}$.
- $\|\lambda_{n,m}\|=\lambda_{1m_1}+\lambda_{2m_2}+...+\lambda_{nm_n}$

For $r,t\in\mathcal{R}^{+^n}$, we define

- $r\leq t$ if and only if $r_i\leq t_i$ and
- $r<t$ if and only if $r_i<t_i$ for $i=1,2,...,n$.

Definition 1.1 :A multiple entire Dirichlet Series is of the form

$$f(s)=\sum_{\|m\|=1}^{\infty}a_{m_1,...,m_n}e^{s.\lambda_{n,m}} \tag{1.1}$$

where $a_{m_1,...,m_n}\in\mathcal{C}$, $s=(s_1,s_2,.....,s_n)\in\mathcal{C}^n$, $s_j=\sigma_j+it_j,j=1,2,...,n$ and $\{\lambda_{j,m_j}\}_{m_j=1}^{\infty}$, $j=1,...,n$ are n sequences of exponents satisfying the conditions

$$0\leq\lambda_{jm_1}<\lambda_{jm_2}<...<\lambda_{jm_k}\to\infty\ \ as\quad k\to\infty,\ j=1,...,n,$$

$\lim\limits_{m_j\to\infty}\dfrac{\log m_j}{\lambda_{jm_j}}=0,j=1,2,...,n.$ and

$$\limsup_{\|m\|\to\infty}\frac{\log|a_{m_1....m_n}|}{\|\lambda_{n,m}\|}=-\infty \tag{1.2}$$

Let $D\subset\mathcal{C}^n$ be an arbitrary complete n-half-plane defined by

$$D=\{s:s\in\mathcal{C}^n,Re(s_i)\leq r_i\} \tag{1.3}$$

where $r = (r_1, r_2, ..., r_n) \in \mathcal{R}^n$. Consider a parameter $R \in \mathcal{R}$, define

$$R + D = D + R = \{s + R : s \in D\} \tag{1.4}$$

for the multiple Dirichlet entire function f, the maximum modulus function $M_{f,D}(R)$ with respect to the region D and $R \in \mathcal{R}$ is defined as

$$M_{f,D}(R) = sup\{\mid f(s) \mid : s \in D + R\} \tag{1.5}$$

$M_{f,D}(R)$ is strictly increasing, increases to ∞ and continuous functions of R. The inverse function is

$$M_{f,D}^{-1} : (L, \infty) \rightarrow (-\infty, \infty)$$

where $0 \leq L = \lim\limits_{R \rightarrow -\infty} M_{f,D}(R)$

Definition 1.2 [1] : The Gol'dberg order of a multiple entire Dirichlet Series f with respect to the domain D is defined by

$$\rho_f(D) = \limsup_{R \rightarrow \infty} \frac{\log \log M_{f,D}(R)}{R} \tag{1.6}$$

Definition 1.3 [1] : The Gol'dberg type of a multiple entire Dirichlet Series f with order $\rho_f(D)$, $(0 < \rho_f(D) < \infty)$ with respect to the domain D, is defined by

$$\sigma_f(D) = \limsup_{R \rightarrow \infty} \frac{\log M_{f,D}(R)}{e^{R\rho_f(D)}} \tag{1.7}$$

To refine the growth of functions whose orders are same but are of infinite type, the concept of proximate order was introduced by G. Valiron [3]. Proximate order is considered as the intermediate comparison function.

Remark 1.1 ([1] p. 64) : established the basic fact that every entire function has a proximate order. Therefore we do not need to prove the existence of proximate Gol'dberg order or proximate Gol'dberg type of a multiple entire Dirichlet series.

2. Proximate Gol'dberg Order of a Multiple Entire Dirichlet Series

Here we define:

Definition 2.1 : A positive continuous function $\rho_D(R)$, satisfying the following properties:

1. $\rho_D(R)$ is differentiable for all large R except for some isolated points where $\rho'_D(R-0)$ and $\rho'_D(R+0)$ exists.
2. $\lim\limits_{R\to\infty} \rho_D(R) = \rho_f$
3. $\lim\limits_{R\to\infty} R\rho'_D(R) = 0$
4. $\limsup\limits_{R\to\infty} \dfrac{\log M_{f,D}(R)}{e^{R\rho_D(R)}} = 1,$

is a proximate Gol'dberg order of a multiple entire Dirichlet series f .

Note : Although the Gol'dberg order ρ_f of f is independent of the choice of the complete n half plane D, proximate Gol'dberg order $\rho_D(R)$ of f is dependent on D. We show this by the following example.

Example 2.1 : Let

$$f(s_1, s_2) = e^{\alpha e^{\beta(s_1+s_2)}} = 1 + \alpha e^{\beta(s_1+s_2)} + \frac{(\alpha e^{\beta(s_1+s_2)})^2}{2!} + ...\infty$$

be a multiple entire Dirichlet series where $\beta > 0$ and $\alpha \in \mathcal{C}$.

For the complete 2-half plane $D = \{s : s \in \mathcal{C}^2, Re(s_i) \le r_i\}$ where $r = (r_1, r_2) \in \mathcal{R}^2$, we have,

$$M_{f,D}(R) = sup\{| f(s) |: s \in D + R\}$$
$$= e^{|\alpha| e^{\beta(r_1+r_2+2R)}}$$

The Gol'dberg order is

$$\rho_f = \limsup_{R\to\infty} \frac{\log\log M_{f,D}(R)}{R}$$

$$= \limsup_{R\to\infty} \frac{\log(|\alpha| e^{\beta(r_1+r_2+2R)})}{R}$$

$$= \limsup_{R\to\infty} \frac{\log|\alpha| + \beta(r_1+r_2+2R)}{R} = 2\beta \qquad (2.8)$$

A proximate Gol'dberg order of f can be taken as:

$$\rho_D(R) = \frac{\log|\alpha|}{R} + \frac{\beta(r_1+r_2)}{R} + 2\beta; \quad R \neq 0$$

Now we check all the properties of proximate Gol'dberg order for $\rho_D(R)$

1. $\rho_D(R)$ is real, continuous and piecewise differentiable for $R > R_0$, since $\frac{1}{R}$ is real, continuous and differentiable for $R \neq 0$

2. $\lim_{R\to\infty} \rho_D(R) = \lim_{R\to\infty} (\frac{\log|\alpha|}{R} + \frac{\beta(r_1+r_2)}{R} + 2\beta) = 2\beta = \rho_f$

3. $\lim_{R\to\infty} R\rho'_D(R) = \lim_{R\to\infty} R[\frac{-\log|\alpha|}{R^2} - \frac{\beta(r_1+r_2)}{R^2}] = 0$

4. $\lim_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{R\rho_D(R)}} = \lim_{R\to\infty} \frac{|\alpha|e^{\beta(r_1+r_2+2R)}}{e^{R\rho_D(R)}} = \lim_{R\to\infty} \frac{|\alpha|e^{\beta(r_1+r_2+2R)}}{e^{R(\frac{\log|\alpha|}{R}+\frac{\beta(r_1+r_2)}{R}+2\beta)}}$
$= \lim_{R\to\infty} \frac{|\alpha|e^{\beta(r_1+r_2+2R)}}{e^{\log|\alpha|}.e^{\beta(r_1+r_2+2R)}} = 1$

Hence $\rho_D(R)$ is a proximate Gol'dberg order of f. Here $\rho_D(R)$ is a function of r_1 and r_2 and hence $\rho_D(R)$ depends on D.

We know that proximate Gol'dberg order is not a unique function, it is the function which approximates the value of Gol'dberg order by satisfying properties as described in Definition 2.1. In the next theorem we have established a method to construct a new proximate Gol'dberg order from existing one.

Theorem 2.1 : Let f be a multiple Dirichlet entire function with finite Gol'dberg order ρ_f . If $\rho_D(R)$ is a proximate Gol'dberg order of f with respect to the arbitrary complete n-half-plane D defined in (1.3), then

$$\rho^*_D(R) = \rho_D(R) + \frac{1}{R^{\rho_f+c}}$$

is a proximate Gol'dberg order of f, where c is a real constant such that $c > 1 - \rho_f$.

Proof : **(i)** By definition of proximate Gol'dberg order $\rho_D(R)$ is real, continuous and piecewise differentiable for $R > R_0$, therefore $\rho^*_D(R)$ is also so as $\frac{1}{R^{\rho_f+c}}$ is real, continuous and differentiable for $R > 0$ and $c > 1 - \rho_f$.

(ii) $\lim_{R\to\infty} \rho^*_D(R) = \lim_{R\to\infty} (\rho_D(R) + \frac{1}{R^{\rho_f+c}})$

$$= \lim_{R\to\infty} \rho_D(R) + 0 = \rho_f$$

[Since ρ_f and c both are real constants.]

(iii) $\lim_{R\to\infty} R\rho^{*'}_D(R) = \lim_{R\to\infty} R[\rho'_D(R) - \frac{\rho_f+c}{R^{\rho_f+c+1}}]$

$$= \lim_{R\to\infty} R\rho_D^{'}(R) = 0$$

[Since $\rho_D(R)$ is proximate Gol'dberg order]

(iv) $$\lim_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{R\rho_D^*(R)}} = \lim_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{R(\rho_D(R)+\frac{1}{R^{\rho_f+c}})}}$$

$$= \lim_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{R\rho_D(R)}} . \lim_{R\to\infty} \frac{1}{e^{\frac{1}{R^{\rho_f+c-1}}}}$$

$$= 1$$

[Since $c > 1 - \rho_f$ and $\rho_D(R)$ is proximate Gol'dberg order.]

Therefore $\rho_D^*(R)$ satisfies all the properties of proximate Gol'dberg order and hence is a proximate Gol'dberg order of f in several complex variables. □

3. Proximate Gol'dberg Type of a Multiple Entire Dirichlet Series

We define:

Definition 3.1 : A positive continuous function $\sigma_D(R)$, satisfying the following properties:

1. $\sigma_D(R)$ is differentiable for all large R except for some isolated points where $\sigma_D'(R-0)$ and $\sigma_D'(R+0)$ exists.
2. $\lim_{R\to\infty} \sigma_D(R) = \sigma_f(D)$
3. $\lim_{R\to\infty} R\sigma_D'(R) = 0$
4. $\limsup_{R\to\infty} \frac{M_{f,D}(R)}{e^{\sigma_D(R)e^{R\rho_f}}} = 1,$

is called a proximate Gol'dberg type of a multiple entire Dirichlet series f with finite Gol'dberg order ρ_f and finite Gol'dberg type $\sigma_f(D)$ with respect to domain D as defined in (1.3).

A proximate Gol'dberg type is not unique function, it is the function which approximates the value of Gol'dberg type by satisfying properties as given in Definition (3.1). In the following theorem, we have found a method to construct a new proximate Gol'dberg type with the help of existing one.

Theorem 3.1 : Let f be a multiple entire Dirichlet series with finite non-zero Gol'dberg order ρ_f, finite Gol'dberg type $\sigma_f(D)$ and proximate Gol'dberg type $\sigma_{_D}(R)$ with respect to the arbitrary complete n-half-plane D, defined in (**??**). For a real differentiable function $\phi(R)$ satisfying the conditions

1. $\lim_{R\to\infty} \phi(R) = 0$
2. $\lim_{R\to\infty} R\phi'(R) = 0$
3. $\limsup_{R\to\infty} \phi(R)e^{R\rho} = 0,$

the function $\sigma^*_{_D}(R) = \sigma_{_D}(R) + \phi(R)$ also will be a proximate Gol'dberg type of f.

Proof : **(i)** By definition of proximate Gol'dberg order $\sigma_{_D}(R)$ is real, continuous and piecewise differentiable for $R > R_0$. Therefore $\sigma^*_{_D}(R)$ is also so as $\phi(R)$ is real, continuous and differentiable.

(ii) $$\lim_{R\to\infty} \sigma^*_{_D}(R) = \lim_{R\to\infty} (\sigma_D(R) + \phi(R))$$

$$= \lim_{R\to\infty} \sigma_D(R) + 0 = \sigma_f(D)$$

(iii) $$\lim_{R\to\infty} R\sigma^{*'}_D(R) = \lim_{R\to\infty} R[\sigma'_D(R) + \phi'(R)]$$

$$= \lim_{R\to\infty} R\sigma'_D(R) = 0.$$

[Since $\lim_{R\to\infty} R\phi'(R) = 0$ and $\sigma_D(R)$ is proximate Gol'dberg type]

(iv) $$\lim_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{\sigma^*_D(R)e^{R\rho_f}}} = \lim_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{(\sigma_D(R)+\phi(R))e^{R\rho_f}}}$$
$$= \lim_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{\sigma_D(R)e^{R\rho_f}}} . \lim_{R\to\infty} \frac{1}{e^{\phi(R)e^{R\rho_f}}} = 1$$

[Since $\lim_{R\to\infty} \phi(R)e^{R\rho_f} = 0$ and $\sigma_{_D}(R)$ is proximate Gol'dberg order.]

Therefore $\sigma^*_D(R)$ satisfies all the properties of proximate Gol'dberg type and hence is a proximate Gol'dberg type of f in several complex variables. □

Example 3.1 : $\phi(R)$ can be taken as $\frac{1}{e^{kR}}$ where $k(>\rho_f)$ is a real number. Whereas $\phi(R)$ can not be of the form $\frac{1}{R^k}$, where k is a real number or $\frac{1}{\log R}$ or $\frac{1}{\log\log R}$.

Example 3.2 : Let $f(s) = e^{\alpha e^{\beta(s_1+s_2)}}$ be a multiple entire Dirichlet series where $\beta > 0$ and $\alpha \in \mathcal{C}$. Also let D be a complete 2-half plane $\{s : s \in \mathcal{C}^2, Re(s_i) \leq r_i\}$ where $r = (r_1, r_2) \in \mathcal{R}^2$. We show that Gol'dberg type $\sigma_f(D)$ and proximate Gol'dberg type $\sigma_D(R)$ of f with respect to the domain D is dependent upon D, although the Gol'dberg order ρ_f of f is independent of D.

$$\text{We have,} \quad M_{f,D}(R) = sup\{\mid f(s) \mid : s \in D + R\}$$
$$= e^{|\alpha| e^{\beta(r_1+r_2+2R)}}$$

$$\text{The Gol'dberg order } \rho_f = 2\beta \text{ [by (2.8)]}$$

2β is independent of domain D

The Gol'dberg type
$$\sigma_f(D) = \limsup_{R\to\infty} \frac{\log M_{f,D}(R)}{e^{\rho_f R}}$$
$$= \limsup_{R\to\infty} \frac{|\alpha| e^{\beta(r_1+r_2+2R)}}{e^{R(2\beta)}}$$
$$= |\alpha| e^{\beta(r_1+r_2)}$$

A proximate Gol'dberg type can be taken as:
$$\sigma_D(R) = |\alpha| e^{\beta(r_1+r_2)} + \frac{1}{e^{kR}}$$
where $k(> 2\beta)$ is a real number and $R > 0$.

All the properties of proximate Gol'dberg type are satisfied by $\sigma_D(R)$ which is dependent upon the choice of domain D.

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References

[1] Alam Md. Feruj, Gol'dberg order and Gol'dberg type of entire functions represented by multiple Dirichlet series. GANIT J. Bangladesh Math. Soc. (ISSN 1606 - 3694) 29 (2009), 63-70.

[2] Levin B. J., Distribution of Zeros of Entire Functions, American mathematical society providence, Rhode Island, (1964).

[3] Valiron G., Lectures on the General Theory of Integral Functions, Chelsea Publishing Company, New York, (1949).